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SIGN-DEFINITENESS CONDITIONS FOR COMPLEX FUNCTIONS AND THE STABILITY OF THE MOTION OF NON-LINEAR SYSTEMS*

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Methods are developed for constructing sign-constant and sign-definite functions. Theorems are proved on the stability and instability of motion with a composite Lyapunov function, enabling the attraction domains to be estimated and determined. Examples are considered.

The development of the Lyapunov vector function method /1/, the formulation and solution of various problems of the stability of motion for variable subsets /2/, multistability /3/, together with experience in the design of automated systems employing these methods, have shown the need for further developments in methods and algorithms for constructing Lyapunov functions and for obtaining fairly simple and constructive conditions for their sign-definiteness. At present, the necessary and sufficient conditions for sign-definiteness have only been obtained for quadratic forms. These are the well-known Sylvester criterion and the recursive criterion /4/. For higher-order forms and their sums, basically only sufficiently conditions for sign-definiteness have been obtained /4, 5/. Furthermore, many papers on the solution of various problems of the stability of motion determine the sign-definiteness of the functions being used from estimates of their values.

This paper generalizes these approaches to obtain sign-definiteness conditions for composite functions and gives a method for constructing Lyapunov functions in the form of a composition of known sign-definite functions possessing special mapping properties.

1. Construction of sign-constant and sign-definite composite functions. Suppose that in the domain (open set) $G_y \subseteq R^m$ $(0 \in G_y)$ we are given a continuous real Lyapunov function $V\colon G_y \to H_v \subseteq R^1$ of the real variable $\mathbf{y} = (y_1, \ldots, y_m)^T \in R^m$, which can in general be sign-constant or sign-definite in some domain $G_y^\circ \subseteq G_y$ $(0 \in G_y^\circ)$. Here H_v is the image domain of the function $\mathbf{y} \to V(\mathbf{y})$ and R^m is an m-dimensional Euclidean space.

It is known /1/ that the function $\mathbf{y} \to V(\mathbf{y})$ is sign-constant in the domain G_y° if $V(\mathbf{y}) \geqslant 0, \forall \mathbf{y} \in G_y^\circ \setminus 0$ or $V(\mathbf{y}) \leqslant 0, \forall \mathbf{y} \in G_y^\circ \setminus 0$ and $V(0) = 0$. If $V(0) = 0$ and $V(\mathbf{y}) > 0, \forall \mathbf{y} \in G_y^\circ$ or $V(\mathbf{y}) < 0, \forall \mathbf{y} \in G_y^\circ \setminus 0,$ then the function $\mathbf{y} \to V(\mathbf{y})$ is called sign-definite in the domain G_y°.

Suppose further that with the help of the continuous mapping $f\colon G_x^* \to G_y^*$ $(0 \in G_x^* \subseteq R^n,$ $0 \in G_y^* \subseteq R^m)$ of the form

$$\mathbf{y} = \mathbf{f}(\mathbf{x}), \mathbf{f}(0) = 0; \quad \mathbf{x} = (x_1, \ldots, x_n)^T \in R^n, \quad \mathbf{f} = (f_1, \ldots, f_m)^T \tag{1.1}$$

the function $\mathbf{y} \to V(\mathbf{y})$ transforms into the function $W\colon G_x \to H_w \subseteq R^1$ where G_y^* is the image domain and G_x^* the domain of definition of the map $\mathbf{y} = \mathbf{f}(\mathbf{x})$ (1.1), G_x is the domain of definition $(G_x \subseteq G_x^*)$ and H_w is the image domain $(H_w \subseteq H_v)$ of the function $\mathbf{x} \to W(\mathbf{x})$, and R^n is n-dimensional Euclidean space.

We wish to find the properties of the map $\mathbf{y} = \mathbf{f}(\mathbf{x})$ (1.1) for which the function $\mathbf{x} \to W(\mathbf{x})$ possesses the properties of the function $\mathbf{y} \to V(\mathbf{y})$, i.e. that it is either sign-constant or sign-definite in some non-empty domain $G_x^\circ \subseteq R^n$ $(0 \in G_x^\circ)$, depending on which properties are possessed by the function $\mathbf{y} \to V(\mathbf{y})$ in the domain G_y°. We denote by $G_y^{*\circ}$ the domain of images of all $\mathbf{x} \in G_x^\circ$ under the map $\mathbf{y} = \mathbf{f}(\mathbf{x})$ (1.1), i.e. the image of G_x°.

Definition 1. The continuous map $\mathbf{y} = \mathbf{f}(\mathbf{x})$ (1.1) is constantly non-trivial if $\mathbf{f}(0) = 0$ and there are values $\mathbf{x} \in G_x^* \setminus 0$ such that $\mathbf{y} = \mathbf{f}(\mathbf{x}) \neq 0$.

Definition 2. The continous map $\mathbf{y} = \mathbf{f}(\mathbf{x})$ (1.1) is definitely non-trivial in the domain G_x° $(0 \in G_x \subseteq G_x^*)$ if $\mathbf{f}(0) = 0$ and for all $\mathbf{x} \in G_x^\circ \setminus 0$ the corresponding $\mathbf{y} = \mathbf{f}(\mathbf{x}) \neq 0$.

We note that the set of constantly non-trivial maps includes the definitely non-trivial maps. This is similar to the inclusion of the sign-definite Lyapunov functions in the set of sign-constant functions.

Theorem 1. If the continuous function $\mathbf{y} \to V(\mathbf{y})$ is sign-constant in the domain $G_y^\circ \subseteq R^m$ and the map $y = f(x)$ (1.1) is constantly non-trivial and $G_y^{*\circ} \subseteq G_y^* \cap G_y^\circ$, then their

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composition $W(x) = V(f(x))$ is a continuous sign-constant function in the domain $G_x^\circ \subseteq G_x^*$ with the same sign as the original function $y \to V(y)$.

Proof. Suppose that the conditions of the theorem are satisfied, and that, to be specific, we have $V(y) \geqslant 0, \forall y \in G_y^\circ \setminus 0$ and $V(0) = 0$. For $x = 0$ we have $y = 0$ and so $W(0) = V(0) = 0$. We then choose an aribitrary point $x \in G_x^\circ \setminus 0$ in the domain $G_x^\circ \subseteq G_x^*$. Under the map $y = f(x)$ (1.1) this point transforms into a point $y \in G_y^{*\circ} \subseteq G_y^* \cap G_y^\circ$ where $V(y) \geqslant 0$. We therefore have the inequality $W(x) \geqslant 0$, $\forall x \in G_x^\circ \setminus 0$ and $W(0) = 0$ for the composition $W(x) = V(f(x))$ at any point $x \in G_x^\circ$. Similarly, for $V(y) \leqslant 0$ we obtain $W(x) \leqslant 0, \forall x \in G_x^\circ \setminus 0, W(0) = 0$. Thus, the function $x \to W(x)$ is sign-constant in the domain G_x° with the same sign as the function $y \to V(y)$ in the domain G_y°. The continuity of the function $x \to W(x)$ follows from the continuity of the function $y \to W(y)$ and the continuity of the map $y = f(x)$ (1.1).

We note that if the function $y \to V(y)$ and the map $y = f(x)$ are differentiable, then the composition is also differentiable. We shall frequently use these properties of composite functions in stability investigations.

Example 1. The function $V(y) = y_1^2 + (y_2 - y_3)^2$ is differentiable and sign-constant throughout the space R^3, and the map $y_1 = x_1 + \operatorname{tg} x_2, y_2 = \sin x_2, y_3 = x_1 \cos x_2$ is differentiable and constantly non-trivial in the domain $G_x^* = \{x_1, x_2): |x_1| < \pi/2, x_2 \in R^1\}$. All the conditions of Theorem 1 are satisfied. Hence the function $W(x) = V(y(x)) = (x_1 + \operatorname{tg} x_2)^2 + (\sin x_2 - x_1 \cos x_2)^2$ is differentiable and sign-constant in the domain $G_x^\circ = G_x^*$ with the same sign as the given function $y \to V(y)$.

Corollary 1. The linear map $y = Ax$, where A is a real $m \times n$ matrix, preserves constant positivity of functions in R^n.

Theorem 2. If the continuous function $y \to V(y)$ is sign-definite in the domain $G_y^\circ \subseteq R^m$, and the map $y = f(x)$ (1.1) is definitely non-trivial in the domain $G_x^\circ \subseteq G_x^*$ with $G_y^{*\circ} \subseteq G_y^* \cap G_y^\circ$, then their composition $W(x) = V(f(x))$ is a continuous sign-definite function in the domain G_x° with the same sign as the original function $y \to V(y)$.

Proof. Suppose the conditions of the theorem are satisfied. To fix our ideas we take $V(y) > 0$, $\forall y \in G_y^\circ \setminus 0$, $V(0) = 0$. It follows from the conditions of the theorem that $W(0) = V(0) = 0$. We now choose an arbitrary point $x \in G_x^\circ \setminus 0$ $(G_x^\circ \subseteq G_x^*)$. The map $y = f(x)$ (1.1) takes this point x into the point $y \in G_y^{*\circ} \subseteq G_y^\circ \cap G_y^*$. From the definite non-triviality of the map $y = f(x)$ we obtain $y \neq 0$ and consequently $V(y) > 0$.

Thus the composition $W(x) = V(f(x))$ at an arbitrary point $x \in G_x^\circ$ satisfies the condition $W(x) > 0$, $\forall x \in G_x^\circ \setminus 0$, $W(0) = 0$. Similarly, for $V(y) < 0$, $\forall y \in G_y^\circ \setminus 0$ we obtain $W(x) < 0$, $\forall x \in G_x^\circ \setminus 0$, $W(0) = 0$. The function $x \to W(x)$ is therefore sign-definite in the domain G_x° with the same sign as the function $y \to V(y)$ in the domain G_y°. The continuity of the function $x \to W(x)$ follows from the continuity of the function $y \to V(y)$ and the continuity of the map $y = f(x)$ (1.1).

Example 2. The function $V(y) = y_1^2 - y_1 y_2 + y_2^2$ is continuous and positive definite throughout the space R^2, and the map $f: R^2 \to R^2$ of the form $y_1 = x_1$, $y_2 = x_2 \cos x_1$ is definitely non-trivial in the space R^2. All the conditions of Theorem 2 are satisfied. Hence the function $W(x) = x_1^2 - x_1 x_2 \cos x_1 + x_2^2 \cos^2 x_1$ is continuous and positive definite in R^2.

The map $f: R^3 \to R^2$ of the form $y_1 = \sqrt{x_1^2 + x_3^2}$, $y_2 = x_2 \cos x_1$ is continuous and definitely non-trivial in R^3. Hence the function $W(x) = x_1^2 + x_3^2 - x_2\sqrt{x_1^2 + x_3^2} \times \cos x_1 + x_2^2 \cos^2 x_1$ is continuous and positive definite in R^3.

The map $f: R^2 \to R^1$ of the form $y_1 = x$, $y_2 = ax$, $a \neq 0$ is also continuous and definitely non-trivial in R^1. Consequently, the function $W(x) = (1 - a + a^2) x^2$ is continuous and positive definite in R^1.

Corollary 2. The non-degenerate linear transformation $y = Ax$ where A is an $n \times n$ real matrix and $\det A \neq 0$, preserves the positive definiteness of functions defined in R^n.

Remark. No restrictions are imposed on the dimensionalities m and n of the mapped spaces. Hence in certain special cases a sign-constant function can become sign-definite, and, conversely, a sign-definite function can become sign-constant. For example, if we apply the definitely non-trivial map $y_1 = x_1$, $y_2 = x_1 + x_2$, $y_3 = x_1$ to the sign-constant function $V(y) = y_1^2 + (y_2 - y_3)^2$, we obtain the sign-definite function $V(x) = x_1^2 + x_2^2$ in the space R^2. Going the other way, we obtain a sign-constant function from a sign-definite function in R^3. Here the inverse map $x_1 = y_1$, $x_2 = y_2 - y_3$ is already constantly non-trivial, because for $y_1 = 0$, $y_2 = y_3 \neq 0$ we obtain $x_1 = x_2 = 0$.

2. ***Conditions for sign-definiteness, using quadratic forms.*** We consider the quadratic form

$$V(y) = \sum_{i_1=1}^{n-m} A_{i_1 i_2} y_{i_1} y_{i_2}^* + \sum_{i_1=n-m+1}^{n} A_{i_1 i_1} y_{i_1}^2 + \sum_{i_1=1}^{n} \sum_{i_2=2}^{n} A_{i_1 i_2} y_{i_1} y_{i_2} \, (i_1 \neq i_2) \tag{2.1}$$

in which $n-m$ coordinates y_{i_1} $(i_1 = 1, 2, \ldots, n-m)$ are selected and tagged with asterisks. The real numbers $A_{i_1 i_2}$ $(i_1, i_2 = 1, \ldots, n)$ form a positive-definite $n \times n$ matrix, i.e. the form (2.1) is positive definite.

We introduce a continuous map $f\colon G_y^f \to G_{y^*}$ (where $0 \in G_y^f \subseteq R^n$ and $0 \in G_{y^*} \subseteq R^{n-m}$) of the form

$$y_{i_1}^* = \mathrm{f}(y_1, \ldots, y_m),\ f(0) = 0,\ i_1 = 1, 2, \ldots, n-m \tag{2.2}$$

where G_y^f is the image domain and G_{y^*} the domain of definition of the map $\mathbf{y}^* = \mathbf{f}(\mathbf{y})$ (2.2), $\mathbf{y}^* = (y_1, y_2, \ldots, y_{n-m}) \in R^{n-m}$, $\mathbf{f} = (f_1, \ldots, f_{n-m})$ is a vector-function, $\mathbf{y} = (y_1, \ldots, y_n) \in R^n$ and R^{n-m} is an $(n-m)$-dimensional Euclidean space.

Using this map as a substitution, we transform the quadratic form $V(\mathbf{y})$ (2.1) into the function $W\colon G_y \to H_w \subseteq R^1$, i.e.

$$W(\mathbf{y}) = \sum_{i_1=1}^{n-m} A_{i_1 i_1} y_{i_1} f_{i_1}(y_1, \ldots, y_n) + \sum_{i_1=n-m+1}^{n} A_{i_1 i_1} y_{i_1}^2 + \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} A_{i_1 i_2} y_{i_1} y_{i_2} (i_1 \neq i_2) \tag{2.3}$$

where G_y is the domain of definition and H_w the image domain of the function $\mathbf{y} \to W(\mathbf{y})$ (2.3).

We wish to specify the properties of the map $\mathbf{y}^* = \mathbf{f}(\mathbf{y})$ (2.2) for which the function $\mathbf{y} \to W(\mathbf{y})$ (2.3) is positive definite in some non-empty domain $G_y^\circ \subseteq R^n$ $(0 \in G_y^\circ \subseteq G_y)$.

Theorem 3. If the quadratic form $V(\mathbf{y})$ (2.1) is positive definite, and the map $\mathbf{y}^* = \mathbf{f}(\mathbf{y})$ (2.2) is definitely non-trivial in the domain $G_y^\circ \subseteq G_y^f$ and satisfies the inequalities

$$f_{i_1}(\mathbf{y})/y_{i_1} \geqslant 1, \quad \forall i_1 = 1, 2, \ldots, n-m; \quad \forall \mathbf{y} \in G_y^\circ \tag{2.4}$$

then the function $\mathbf{y} \to W(\mathbf{y})$ is positive definite in the domain $G_y^\circ \cap R^m$.

Proof. Suppose the conditions of the theorem are satisfied. Then inequalities (2.4) imply that

$$A_{11} y_1 f_1(y) - A_{11} y_1^2 \geqslant 0\ (A_{11} > 0), \ldots, A_{n-m,\,n-m}\, y_{n-m} f_{n-m}(\mathbf{y}) - A_{n-m,\,n-m}\, y_{n-m}^2 \geqslant 0\ (A_{n-m,\,n-m} > 0) \tag{2.5}$$

Summing the $n-m$ non-negative functions (2.5) and the positive definite quadratic form $V(\mathbf{y})$ (2.1), we obtain the function $\mathbf{y} \to W(\mathbf{y})$ (2.3), which will be positive definite in the domain $G_y^\circ \cap R^m$. Indeed, at the point $y = 0$ we have $W(0) = 0$ (2.3), while for any point $\mathbf{y} \in G_y^\circ \setminus 0$ we have $W(\mathbf{y}) > 0$, because at that point $\mathbf{y}$ we are adding together $V(\mathbf{y}) > 0$ and the sum of non-negative quantities (2.5). The theorem is proved.

Example 3. The function $F(\mathbf{x}) = ax_1 \sin x_1 + x_1 x_2 + x_2^2$ will be positive definite in the domain $G_x^\circ = \{(x_1, x_2)\colon |x_1| < \pi/2,\ x_2 \in R^1\}$ for $a \geqslant \pi/2$. This follows from condition (2.4) of Theorem 3, i.e. the inequality $a \sin x_1/x_1 \geqslant 1$ for $|x_1| < \pi/2$ is satisfied if $a \geqslant \pi/2$.

Theorem 4. If $A_{i_1 i_2}$ $(i_1, i_2 = 1, \ldots, m)$ are real numbers forming a symmetric $m \times m$-matrix, then for the function

$$W(\mathbf{x}) = \sum_{i_1=1}^{m-p} A_{i_1 i_1} f_{i_1}(x) F_{i_1}(f(x)) + \sum_{i_1=m-p+1}^{m} A_{i_1 i_1} f_{i_1}^2(x) + \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} A_{i_1 i_2} f_{i_1}(x) f_{i_2}(x) (i_1 \neq i_2), \quad W(0) = 0 \tag{2.6}$$

to be positive definite in the domain $G_x^\circ \subseteq R^n$ it is sufficient for there to exist real numbers

$$a_{ij} = \frac{1}{a_{ii}}\Big(A_{ij} - \sum_{k=1}^{i-1} a_{ki} a_{kj}\Big) \tag{2.7}$$

$$i = 1, 2, \ldots, m;\ j = i, i+1, \ldots, m;\ i > k \geqslant 1,\ a_{ki} = 0,\ \forall k \geqslant i$$

satisfying the condition

$$a_{ii} = 0,\ \forall i = 1, \ldots, m, \tag{2.8}$$

a map $F\colon G_y^F \to G_{y^*} \subseteq R^{m-p}$ for the selected coordinates $y_i^* \in R^{m-p}$:

$$y_{i_1}^* = F_{i_1}(y_1, \ldots, y_m),\ i_1 = 1, 2, \ldots, m-p \tag{2.9}$$

definitely non-trivial in the domain $G_y^\circ \subseteq G_y^F$ and satisfying the inequalities

$$F_{i_1}(\mathbf{y})/y_{i_1} \geqslant 1,\ \forall i_1 = 1, 2, \ldots, m-p, \tag{2.10}$$

and a map $f\colon G_x^f \to G_y^f \subseteq R^m$, definitely non-trivial in the domain $G_x^\circ \subseteq G_x^f \subseteq R^n$, of the form

$$y_{i_1} = f_{i_1}(\mathbf{x}), \quad i_1 = 1, \ldots, m\,. \tag{2.11}$$

Proof. Suppose real numbers a_{ij} (2.7) exist obeying condition (2.8), i.e. the recursive criterion for the sign-definiteness of quadratic forms is obeyed /3/:

$$V(\mathbf{y}) = \sum_{i_1=1}^{m}\sum_{i_2=1}^{m} A_{i_1i_2} y_{i_1} y_{i_2}, \quad A_{i_1i_2} = A_{i_2i_1} \tag{2.12}$$

The recursive formulae (2.7) and condition (2.8) are in fact obtained from the following equality:

$$\sum_{i_1=1}^{m}\sum_{i_2=1}^{m} A_{i_1i_2} y_{i_1} y_{i_2} = \sum_{i=1}^{m}\Big(\sum_{j=i}^{m} a_{ij} y_j\Big)^2 \tag{2.13}$$

where for condition (2.8) the right-hand side is a positive-definite quadratic form, while the left is the quadratic form $V(\mathbf{y})$ (2.12). Hence the form $V(\mathbf{y})$ (2.12) is also positive definite.

Suppose there exists a map $F\colon G_y^F \to G_y^*$ (2.9), definitely non-trivial in the domain $G_y^\circ \subseteq G_y^f \subseteq G_y^F$, acting only on the selected coordinates $y_{i_1}^*$ $(i_1 = 1, \ldots, m-p)$ and satisfying condition (2.10). Then according to Theorem 3 the function

$$V^*(\mathbf{y}) = \sum_{i_1=1}^{m-p} A_{i_1i_1} y_{i_1} F_{i_1}(y_1, \ldots, y_m) + \sum_{i_1=m-p+1}^{m} A_{i_1i_1} y_{i_1}^2 + \sum_{i_1=1}^{m}\sum_{i_2=1}^{m} A_{i_1i_2} y_{i_1} y_{i_2} \,(i_1 \neq i_2) \tag{2.14}$$

will be positive definite in the domain $G_y^\circ \cap R^m$.

If a map $y_{i_1} = f_{i_1}(\mathbf{x})$ (2.11) exists, definitely non-trivial in the domain $G_x^\circ \subseteq G_x^f$, then substituting the values of $y_{i_1} = f_{i_1}(\mathbf{x})$, $i_1 = 1, \ldots, m$ (2.11) into the function $V^*(\mathbf{y})$ (2.14), we find from Theorem 2 that the function $W(\mathbf{x}) = W^*(\mathbf{f}(\mathbf{x}))$ is positive definite in the domain G_x°. The theorem is proved.

Example 4. The function $W(x) = 5\sin^2 x + 11\cos^2 x + 7x\sin x - 2x\cos x - 5\sin 2x + 2x + 10\sin x - 22\cos x + 11$ is positive definite for $|x| < \pi/2$, because all the conditions of Theorem 6 are satisfied. Indeed, this function is obtained from the positive definite quadratic form $V(\mathbf{y}) = y_1^2 + 2y_1y_2 + y_1y_3 + 5y_2^2 + 2y_2y_1 + 5y_2y_3 + 11y_3^2 + y_3y_1 + 5y_3y_2$ using a mapping a one selected coordinate y_1 in the first term, i.e. $y_1^* = 3\sin y_1$, definitely non-trivial and satisfying condition (2.10) for $|x| < \pi/2$, and the map $y_1 = x$, $y_2 = \sin x$, $y_3 = 1 - \cos x$, definitely non-trivial for $|x| < \pi$.

3. *Stability and instability theorems with a composite Lyapunov function.* Suppose we are given a system of differential equations for perturbed motion

$$d\mathbf{x}/dt = \mathbf{X}(\mathbf{x}), \quad \mathbf{X}(0) = 0, \quad \mathbf{x} = (x_1, \ldots, x_n) \in R^n \tag{3.1}$$

where $\mathbf{X} = (X_1, \ldots, X_n)$ is a vector function such that existence and uniqueness conditions are satisfied for solutions to Eq.(3.1) in the domain $G = \{\mathbf{x}\colon \|x\| < H = \text{const},\ \|x\|^2 = x_1^2 + \ldots + x_n^2\}$. We shall investigate the stability of the unperturbed motion $\mathbf{x} = 0$ of system (3.1).

Theorem 5. Suppose that for system (3.1) there exist:

a function $\mathbf{y} \to V(\mathbf{y})$, differentiable and positive definite in the domain $G_y^\circ \supseteq G_y^f$;

a map $f\colon G_x^f \to G_y^f$ $(0 \in G_x^f \subseteq R^n,\ 0 \in G_y^f \subseteq R^m)$, $\mathbf{y} = \mathbf{f}(\mathbf{x})$, where $\mathbf{y} = (y_1, \ldots, y_m) \in R^m$ and $\mathbf{f} = (f_1, \ldots, f_m)$ is a vector function, differentiable and definitely non-trivial in the domain $G_x^\circ \subseteq G_x^f \subseteq G$; and

a differentiable and constantly non-trival map $g\colon G_x^g \to G_z^g, (0 \in G_x^g \subseteq R^n,\ 0 \in G_z^g \subseteq R^p)$, $\mathbf{z} = \mathbf{g}(\mathbf{x})$ where $\mathbf{z} = (z_1, \ldots, z_p) \in R^p$ and $\mathbf{g} = (g_1, \ldots, g_p)$ is a vector function, such that the total derivative of the composition $V(\mathbf{f}(\mathbf{x}))$ with respect to t, which from system (3.1) is

$$\frac{dV}{dt} = \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\partial V}{\partial y_i}\frac{\partial f_i}{\partial x_j}\frac{dx_j}{dt} = \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\partial V}{\partial y_i}\frac{\partial f_i}{\partial x_j} X_j(\mathbf{x}), \tag{3.2}$$

and using the map $\mathbf{z} = \mathbf{g}(\mathbf{x})$ is transformed into a constantly negative or identically zero function $\mathbf{z} \to W(\mathbf{z})$, i.e. $dV/dt = W(\mathbf{z})$.

Then the unperturbed motion $\mathbf{x} = 0$ of system (3.1) is stable (uniformly with respect to t_0), and all trajectories emerging from the domain G_x° remain in a bounded domain.

Proof. Suppose the conditions of the theorem are satisfied. Then according to Theorem 2 the composition of the function $y \to V(y)$ which is positive definite in the domain G_y° and the map $y = f(x)$ which is definitely non-trivial in the domain G_x°, i.e. the function $x \to V(f(x))$, is positive definite in the domain $G_x^\circ \subseteq R^n$. According to Theorem 1, the composition of a sign-constant (constantly negative) or identically zero function $z \to W(z)$ and a constantly non-trivial map $z = g(x)$, i.e., the function $x \to W(g(x))$, will be respectively a constantly negative or identically zero function in a neighbourhood of the unperturbed motion $x = 0$. Then all the conditions of Lyapunov's stability theorem /6/ with Persidskii's addition /7/ are satisfied. Because $V(f(x))|_{t>0} \leqslant V(f(x))|_{t=0}$, and because the composite function $x \to V(f(x))$ is positive definite, the trajectories of motion remain in a bounded domain if they emerge from the domain G_x°. The theorem is proved.

Remark. Theorem 5 remains valid if we take as a Lyapunov function the sum $V(f(x)) + p(x)$, where $x \to p(x)$ is non-negative sign-constant function with $p(0) = 0$.

Theorem 6. Suppose that for system (3.1) there exist:

a function $y \to V(y)$, differentiable and positive definite in a domain $G_y^\circ \supseteq G_y^f$;

a map f: $G_x^f \to G_y^f$ $(0 \in G_x^f \subseteq R^n)$, $0 \in G_y^f \in R^m)$, $y = f(x)$, where $y = (y_1, \ldots, y_m) \in R^m$ and $f = (f_1, \ldots, f_m)$ is a vector function, such that f is differentiable and definitely non-trivial in the domain $G_x^\circ \subseteq G_x^f \subseteq G$; and

a map: g: $G_x^g \to G_z^g$ $(0 \in G_x^g \subseteq R^n$, $0 \in G_z^g \subseteq R^p)$, $z = g(x)$, where $z = (z_1, \ldots, z_p) \in R^p$ and $g = (g_1, \ldots, g_p)$ is a vector function, definitely non-trivial in a domain $G_x^W \subseteq G_x^g \subseteq G$, such that the total derivative dV/dt (3.2) of the composition $V(f(x))$ with respect to t from the system (3.1) can, using the map $z = g(x)$, i.e. $dV/dt = W(z)$, be transformed into a negatively definite function $z \to W(z)$ in the domain $G_z^\circ \supseteq G_z^g$.

Then the unperturbed motion $x = 0$ of system (3.1) is asymptotically stable (uniformly with respect to x_0, t_0) and the bounded domain

$$G_{v<c} = \{x : V(f(x)) \leqslant c = \text{const} > 0\} \subseteq G_x^\circ \cap G_x^W \tag{3.3}$$

lies in the attraction domain of the unperturbed motion $x = 0$ of system (3.1).

Proof. Suppose the conditions of the theorem are satisifed. Then according to Theorem 2, the composition of the function $y \to V(y)$ which is positive definite in the domain G_y° and the map $y = f(x)$ which is definitely non-trivial in the domain G_x°, i.e., the function $x \to V(f(x))$, is positive definite in the domain G_z°. According to this theorem, the composition of a function $z \to W(z)$ that is negative definite in the domain G_z° and a map $z = g(x)$ that is definitely non-trivial in the domain G_x^W, i.e. the function $x \to W(g(x))$, will be negative definite in the domain G_x^W $(0 \in G_x^W \cap G_x^\circ \neq \varnothing)$. Then all the asymptotic stability conditions of Lyapunov's theorem /6/ with Malkin's addition /7/ are satisfied and the unperturbed motion $x = 0$ of system (3.1) is asymptotically stable, uniformly with respect to x_0 and t_0.

Here the bounded domain $G_{v<c}$ (3.3) is contained in the intersection of the domains G_x° and G_x^W, and so the level surfaces of the positive definite function $x \to V(f(x))$ are closed surfaces nested inside one another. Hence any trajectory $x(t; x_0, t_0)$ with initial value $x_0 = x(t_0) \in G_{v<c}$ will cross the surfaces from the outside to the inside and tend to the solution $x = 0$ as $t \to \infty$. The theorem is proved.

The remark for Theorem 5 also applies to Theorem 6.

Theorem 7. Suppose that for system (3.1) there exist:

a differentiable function $y \to V(y)$ taking positive values at some points y in any arbitrarily small neighbourhood G_y^ε of the origin of coordinates $\{y = 0\} \in R^m$;

a map f: $G_x^f \to G_y^f$ $(0 \in G_x^f \subseteq R^n$, $0 \in G_y^f \subseteq R^m)$, where $y = f(x)$, $y = (y_1, \ldots, y_m) \in R^m$ and $f = (f_1, \ldots, f_m)$ is a vector function, differentiable and definitely non-trivial in the domain $G_x^\circ \subseteq G_x^f \subseteq G$; and

a map g: $G_x^g \to G_z^g$, $z = g(x)$ $(0 \in G_x^f \subseteq R^n$, $0 \in G_y^f \subseteq R^m)$, where $z = (z_1, \ldots, z_p) \in R^p$ and $g = (g_1, \ldots, g_p)$ is a vector function, definitely non-trivial in the domain $G_x^W \subseteq G_x^g \subseteq G$, such that the total derivative dV/dt (3.2) of the composition $x \to V(f(x))$ with respect to t from (3.1) can, using the map $z = g(x)$, be transformed into a positive definite function $z \to W(z)$ in the domain $G_z^\circ \subseteq G_z^g$, i.e. $dV/dt = W(z)$.

Then the unperturbed motion $x = 0$ of system (3.1) is unstable.

Proof. Suppose the conditions of the theorem are satisfied. Then, using the continuity of the map $y = f(x)$ at the point $x = 0$, for any arbitrarily small $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\|x\| < \delta$ we have $\|y\| < \varepsilon$. In other words, points $x \in G_x^\delta = \{x: \|x\| < \delta = \text{const} > 0\}$ can correspond to an arbitrarily small neighbourhood of G_y^ε of the origin of coordinates $\{y = 0\} \in R^m$. Because by the conditions of the theorem the function $y \to V(y)$ takes positive values at some points $y \in G_y^\varepsilon$, then by virtue of the non-trivial definiteness of the map $y = f(x)$ the composition $V(f(x))$ is also positive at the corresponding points $x \in G_x^\delta$.

On the other hand, according to Theorem 2, the composition of the positive definite function $z \to W(z)$ in the domain $G_z^\circ \subseteq G_z^g$ and the non-trivially definite map $z = g(x)$ in the domain G_x^W, i.e., the function $x \to W(g(x))$, will be positive definite in the domain G_x^W $(0 \in G_x^W \cap G_x^\circ \neq \varnothing)$.

In this case all the conditions of Lyapunov's first instability theorem /6/ are satisfied. The theorem is proved.

4. Examples. 1) We will apply our results to the stability of the rotational motion of a shell. For a very shallow firing trajectory the following differential equations describe the perturbed motion /8/:

$$\frac{dx_1}{dt} = 2x_1x_2\frac{\sin x_4}{\cos x_4} - \frac{Ap}{B}\frac{x_2}{\cos x_4} + \frac{a}{B}\frac{\sin x_3}{\cos x_4} \tag{4.1}$$

$$\frac{dx_2}{dt} = -x_1^2 \sin x_4 \cos x_4 + \frac{Ap}{B} x_1 \cos x_4 + \frac{a}{B} \sin x_4 \cos x_3$$

$$\frac{dx_3}{dt} = x_1, \qquad \frac{dx_4}{dt} = x_2$$

where x_3 is the angle made by the axis of the shell with its projection onto the firing plane, x_4 is the angle between this projection and the tangent to the trajectory of the centre of mass, and A, B, p and a are constants depending on the parameters and conditions of motion of the shell.

The local stability of the unperturbed motion $x_1 = x_2 = x_3 = x_4 = 0$ was shown in /8/. Here we shall obtain an estimate for the domain of initial perturbations for which the trajectories remain in a bounded domain, as well as proving stability.

Consider the function

$$V_1(y) = (a_{11}y_1 + a_{14}y_4)^2 + (a_{22}y_2 + a_{23}y_3)^2 + (a_{33}y_3)^2 + (a_{44}y_4)^2 \tag{4.2}$$

$$a_{11}^2 = a_{22}^2 = 1/2BAp,\ a_{11}a_{14} = -a_{22}a_{23} = Ba,\ a_{14}^2 + a_{44}^2 = a_{23}^2 + a_{33}^2 = {}^1/_2 Apa$$

that is positive definite in R^4, and in the domain

$$G_x^\circ = \{x\colon x_i \in R^1,\ i = 1, 2;\ |x_j| < \pi/2,\ j = 3, 4\} \tag{4.3}$$

a definitely non-trivial map $y = f(x)$ of the form

$$y_1 = x_1 \cos x_4,\ y_2 = x_2,\ y_3 = \sin x_3,\ y_4 = \sin x_4 \cos x_3 \tag{4.4}$$

We form a Lyapunov function

$$V(x) = V_1(f(x)) + {}^1/_2 Apa\,(1 - \cos x_3 \cos x_4)^2 \tag{4.5}$$

where $x \to V_1(f(x))$ is the composition of functions (4.2) and (4.4).

The total derivative of the function $x \to V(x)$ (4.5) from system (4.1) is identically zero. According to Theorem 5 and its remark, the unperturbed motion $x = 0$ is stable and trajectories emerging from the domain G_x° (4.3) remain in a bounded domain.

2) We will use our results to derive the sufficient conditions for the asymptotic stability for unperturbed motions $x = 0$ of the following autonomous system, encountered in multifrequency oscillation problems:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} c_{ij} \sin x_j, \quad i = 1, \ldots, n; \quad (x_1, \ldots, x_n) = x \in R^n \tag{4.6}$$

where the c_{ij} $(i, j = 1, \ldots, n)$ are real numbers.

Consider the negative definite function

$$V = \sum_{i=1}^{n} b_i y_i^2, \quad b_i = \text{const} < 0, \quad \forall i = 1, \ldots, n \tag{4.7}$$

in R^n, and the definitely non-trivial map $y = f(x)$ of the form

$$y_i = \sqrt{1 - \cos x_i}, \quad i = 1, \ldots, n \tag{4.8}$$

in the domain $G_x^\circ = \{x\colon |x_i| < \pi;\ i = 1, \ldots, n\}$.

The total derivative of the composition of the function (4.7) and the map (4.8) in system (4.6),

$$\frac{dV}{dt} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial V}{\partial y_j}\frac{\partial y_j}{\partial x_i}\frac{dx_i}{dt} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{b_i}{2}(c_{ij} + c_{ji}) \sin x_i \sin x_j$$

can be transformed into the quadratic form

$$W(z) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} z_i z_j, \quad A_{ij} = A_{ji} = {}^1/_2 b_i (c_{ij} + c_{ji}) \tag{4.9}$$

using a map $z = g(x)$ of the form $z_i = \sin x_i \ (i = 1, \ldots, n)$ that is definitely non-trivial in the domain $G_x^W = G_x^\circ$

Applying the recursive criterion for positive definiteness /3/ to the quadratic form $W(z)$ (4.9), we obtain the following assertion: for the asymptotic stability of unperturbed motion $x = 0$ of system (4.6) with attraction domain $G_{v<c} = \{x: |x_i| < \pi, i = 1, \ldots, n\}$ it is sufficient that there exist real numbers $b_i < 0 \ (i = 1, \ldots, n)$ and

$$a_{ij} = \frac{1}{a_{ii}}\left[\frac{b_i}{2}(c_{ij} + c_{ji}) - \sum_{k=1}^{i-1} a_{ki} a_{kj}\right]$$

$$i = 1, \ldots, n;\ j = i, i + 1, \ldots, n;\ i > k \geqslant 1,\ a_{ki} \equiv 0,\ \forall k \geqslant i$$

satisfying the conditions $a_{ii} \neq 0, \ \forall i = 1, \ldots, m$.

Suppose that these conditions are in fact satisfied. Then the function $y \to V(y)$ (4.7) is negative definite, while the function $z \to W(z)$ (4.9) is positive definite. There also exists a map $y = f(x)$ (4.8) that is definitely non-trivial in a domain G_x° and a map $z = g(x)$ that is definitely non-trivial in a domain G_x^W such that $G_x^\circ = G_x^W = \{x: |x_i| < \pi, i = 1, \ldots, n\}$. In this case the domain $G_{v<c} = \{x: V(f(x)) < c = \text{const} > 0\}$ coincides with the domain G_x°, i.e. $G_{v<c} = G_x^\circ = G_x^W$. Because the phase space of system (4.6) is an n-dimensional torus, $G_{v<c}$ is an attraction domain of the unperturbed motion $x = 0$, because there are no other points in the n-dimensional torus from which asymptotically stable trajectories could emerge. In this case the conditions of Theorem 6 are satisfied. The assertion is proved.

We note that because of the periodicity of the right-hand side of system (4.6) the solutions $x_i = 2k\pi \ (k = 1, 2, \ldots), i = 1, \ldots, n$ will be stable when these conditions are satisfied, whereas solutions $x_i = (2k + 1)\pi, (k = 0, 1, 2, \ldots), i = 1, \ldots, n$ will be unstable. Indeed, in any arbitrarily small neighbourhood of the latter there exist points $x \in G_{v<c}$, where $G_{v<c} = \{x: |x_i| < \pi, i = 1, \ldots, n\}$ is an attraction domain of the solutions $x_i = 2k\pi \ (k = 0, 1, 2, \ldots), i = 1, \ldots, n$.

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